Homework 9
Geometry

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Proposition 0.1 (Exercise 12-4). *Let $V_1, \ldots, V_k, W$ be finite-dimensional real vector spaces. There is a canonical isomorphism*

$$V_1^* \otimes \ldots \otimes V_k^* \otimes W \cong L(V_1, \ldots, V_k; W)$$

Proof. Define $\phi : V_1^* \times \ldots \times V_k^* \times W \to L(V_1, \ldots, V_k; W)$ by

$$\phi(\lambda_1, \ldots, \lambda_k, w)(v_1, \ldots, v_k) = \left(\prod_{i=1}^k \lambda_i(v_i) \right) w$$

Note that $\lambda_i : V_i \to \mathbb{R}$ so the product $\prod_{i=1}^k \lambda_i(v_i)$ is in $\mathbb{R}$. Note that ϕ maps into $L(V_1, \ldots, V_k; W)$ because it depends linearly on each v_i, as each λ_i is linear. We claim that ϕ is multi-linear. First we show linearity in the W-component.

$$\begin{aligned}
\phi(\lambda_1, \ldots, \lambda_k, a_1 w_1 + a_2 w_2)(v_1, \ldots, v_k) &= \left(\prod_{i=1}^k \lambda_i(v_i) \right) (a_1 w_1 + a_2 w_2) \\
&= a_1 \left(\prod_{i=1}^k \lambda_i(v_i) \right) w_1 + a_2 \left(\prod_{i=1}^k \lambda_i(v_i) \right) w_2 \\
&= a_1 \phi(\lambda_1, \ldots, \lambda_k, w_1)(v_1, \ldots, v_k) + a_2 \phi(\lambda_1, \ldots, \lambda_k, w_2)(v_1, \ldots, v_k)
\end{aligned}$$

thus we have linearity in the W-component, that is,

$$\phi(\lambda_1, \ldots, \lambda_k, a_1 w_1 + a_2 w_2) = a_1 \phi(\lambda_1, \ldots, \lambda_k, w_1) + a_2 \phi(\lambda_1, \ldots, \lambda_k, w_2)$$

Now we show linearity in the V_j-th component.

$$\begin{aligned}
\phi(\lambda_1, \ldots, a\lambda_j + b\alpha_j, \ldots, \lambda_k, w)(v_1, \ldots, v_k) &= \lambda_1(v_1) \ldots (a\lambda_j + b\alpha_j)(v_j) \ldots \lambda_k(v_k) w \\
&= \lambda_1(v_1) \ldots a\lambda_j(v_j) \ldots \lambda_k(v_k) w + \lambda_1(v_1) \ldots b\alpha_j(v_j) \ldots \lambda_k(v_k) w \\
&= a\lambda_1(v_1) \ldots \lambda_j(v_j) \ldots \lambda_k(v_k) w + b\lambda_1(v_1) \ldots \alpha_j(v_j) \ldots \lambda_k(v_k) w \\
&= a\phi(\lambda_1, \ldots \lambda_j, \ldots, \lambda_k, w)(v_1, \ldots, v_k) + b\phi(\lambda_1, \ldots, \alpha_j, \ldots, \lambda_k, w)(v_1, \ldots, v_k)
\end{aligned}$$

thus we have linearity in the V_j-th component,

$$\phi(\lambda_1, \dots, a\lambda_j + b\alpha_j, \dots, \lambda_k, w) = a\phi(\lambda_1, \dots, \lambda_j, \dots, \lambda_k, w) + b\phi(\lambda_1, \dots, \alpha_j, \dots, \lambda_k, w)$$

Thus ϕ is multilinear. Now, by the characteristic property of tensor product spaces (Proposition 12.7 in Lee), there is a unique linear map

$$\widetilde{\phi} : V_1^* \otimes \dots \otimes V_k^* \otimes W \to L(V_1, \dots, V_k; W)$$

so that $\widetilde{\phi} \circ \pi = \phi$, where $\pi : V_1^* \times \dots V_k^* \times W \to V_1^* \otimes \dots \otimes V_k^* \otimes W$ is the projection $\pi(\lambda_1, \dots, \lambda_k, w) = \lambda_1 \otimes \dots \otimes \lambda_k \otimes w$. We claim that $\widetilde{\phi}$ is an isomorphism. It is sufficient to show that it has trivial kernel, since it is a linear map bewteen spaces of equal dimension. Suppose that $\lambda_1 \otimes \dots \otimes \lambda_k \otimes w \in \ker \widetilde{\phi}$. Then for all $(v_1, \dots, v_k) \in V_1 \times \dots \times V_k$, we have

$$\phi(\lambda_1, \dots, \lambda_k, w)(v_1, \dots, v_k) = \left(\prod_{i=1}^k \lambda_i(v_i) \right) w = 0$$

For $w \neq 0$, this implies that

$$\prod_{i=1}^k \lambda_i(v_i) = 0$$

for all v_i. Then for $w \neq 0$, we have $\lambda_1 \otimes \dots \otimes \lambda_k = 0$. Thus the product $(\lambda_1 \otimes \dots \otimes \lambda_k) \otimes w$ is always zero. Hence the kernel of $\widetilde{\phi}$ is trivial, so it is injective, so it is an isomorphism. □

Lemma 0.2 (for Exercise 14-1)**.** *Let V be a finite dimensional vector space and $\omega^1, \dots, \omega^k$ be covectors. If $\omega^i = \omega^j$ for some $i \neq j$, then $\omega^1 \wedge \dots \wedge \omega^k = 0$.*

Proof. □

Proposition 0.3 (Exercise 14-1)**.** *Let V be a finite dimensional vector space and $\omega^1, \dots, \omega^k$ be covectors. Then $\omega^1 \wedge \dots \wedge \omega^k = 0$ if and only if $\omega^1, \dots, \omega^k$ are linearly dependent.*

Proof. First suppose that the covectors are linearly dependent. Then we can write ω^k as a linear combination of the others,

$$\omega^k = \sum_{i=1}^{k-1} a_i \omega^i$$

Then

$$\omega^1 \wedge \dots \wedge \omega^k = \omega^1 \wedge \dots \wedge \omega^{k-1} \wedge \sum_{i=1}^{k-1} a_i \omega^i = \sum_{i=1}^{k-1} a_i (\omega^1 \wedge \dots \wedge \omega^{k-1} \wedge \omega^i)$$

We claim that any wedge sum with a repeated covector is zero. We have the formula

$$\omega^1 \wedge \dots \wedge \omega^k (v_1, \dots, v_k) = \det(\omega^j(v_i))$$

So if we have a repeated ω^i, then the determinant on the RHS will be a determinant of a matrix with a repeated column, so the determinant will be zero. Hence

$$\omega^1 \wedge \dots \wedge \omega^k = \sum_{i=1}^{k-1} a_i (\omega^1 \wedge \dots \wedge \omega^{k-1} \wedge \omega^i) = \sum_{i=1}^{k-1} a_i (0) = 0$$

Conversely, suppose that $\omega^1 \wedge \ldots \wedge \omega^k = 0$. Again using the determinant formula,

$$\omega^1 \wedge \ldots \wedge \omega^k(v_1, \ldots, v_k) = \det(\omega^j(v_i))$$

we know that the columns of the matrix with ij-th entry $\omega^j(v_i)$ are linearly dependent, so

$$\begin{bmatrix} \omega^k(v_1) \\ \vdots \\ \omega^k(v_k) \end{bmatrix} = \begin{bmatrix} \omega^1(v_1) \\ \vdots \\ \omega^1(v_k) \end{bmatrix} + \ldots + \begin{bmatrix} \omega^{k-1}(v_1) \\ \vdots \\ \omega^{k-1}(v_k) \end{bmatrix} = \sum_{i=1}^{k-1} a_i \begin{bmatrix} \omega^i(v_1) \\ \vdots \\ \omega^i(v_k) \end{bmatrix}$$

Considering the first row of this matrix equation, we have

$$\omega^k(v_1) = \sum_{i=1}^{k-1} a_i \omega^i(v_i)$$

for any $v_1 \in V$. Thus $\omega^k = \sum_{i=1}^{k-1} a_i \omega^i$, so the covectors are linearly dependent.

$\square$

Proposition 0.4 (Exercise 14-5, Cartan's Lemma). *Let M be a smooth n-manifold with or without boundary and let $(\omega^1, \ldots, \omega^k)$ be an ordered k-tuple of smooth 1-forms on an open subset $U \subset M$ such that $(\omega^1|_p, \ldots, \omega^k|_p)$ is linearly independent for each $p \in U$. Given smooth 1-forms $\alpha^i, \ldots, \alpha^k$ on U such that*

$$\sum_{i=1}^{k} \alpha^i \wedge \omega^i = 0$$

then each α^i can be written as a linear combination of $\omega^1, \ldots, \omega^k$ with smooth coefficients.

Proof. Suppose we have such 1-forms α^i. By linearity of the wedge product, if we wedge anything with zero, we get zero, so

$$\left(\omega^1 \wedge \ldots \wedge \widehat{\omega}^j \wedge \ldots \omega^k\right) \wedge \left(\sum_{i=1}^{k} \alpha^i \wedge \omega^i\right) = \left(\omega^1 \wedge \ldots \wedge \widehat{\omega}^j \wedge \ldots \omega^k\right) \wedge 0 = 0$$

where $\widehat{\omega}^j$ indicates the omission of ω^j from the k-fold wedge product. By linearity, if we expand this,we also get

$$\begin{aligned} \left(\omega^1 \wedge \ldots \wedge \widehat{\omega}^j \wedge \ldots \omega^k\right) \wedge \left(\sum_{i=1}^{k} \alpha^i \wedge \omega^i\right) &= \omega^1 \wedge \ldots \wedge \widehat{\omega}^j \wedge \ldots \wedge \omega^k \wedge \alpha^j \wedge \omega^j \\ &= \pm\, \omega^1 \wedge \ldots \wedge \omega^k \wedge \alpha^j \end{aligned}$$

after some transpositions possibly introducing a $(-1)^n$. Hence

$$\omega^1 \wedge \ldots \wedge \widehat{\omega}^j \wedge \ldots \wedge \omega^k \wedge \alpha^j \wedge \omega^j = 0$$

By Exercise 14-1, this means that $\omega^1, \dots, \omega^k, \alpha^j$ are linearly dependent, so

$$\alpha_j = \sum_{i=1}^{k} a_i^{(j)} \omega^i$$

Since ω^i are all smooth and each α_j is smooth, and the ω^i form a smooth frame on U, the component functions of α^j in this smooth frame must be smooth, using proposition 10.22 (page 260 of Lee). Hence each $a_i^{(j)}$ is smooth. □

Proposition 0.5 (Exercise 14-6a)**.** *Define a 2-form on* $\mathbb{R}^d$ *by*

$$\omega = x\ dy \wedge dz + y\ dz \wedge dx + z\ dx \wedge dy$$

In spherical coordinates $(x, y, z) = (\rho \sin\phi \cos\theta, \rho \sin\phi \sin\theta, \rho \cos\phi)$ *we can rewrite* ω *as*

$$\omega = \rho^3 \sin\phi\ d\phi \wedge d\theta$$

Proof. This is simply a long, arduous, and tedious computation. Expand everything out, collect terms with common wedge products, and apply the trigonometric identity $\sin^2 x + \cos^2 x = 1$ several times. □

Proposition 0.6 (Exercise 14-6b)**.** *Define a 2-form on* $\mathbb{R}^d$ *by*

$$\omega = x\ dy \wedge dz + y\ dz \wedge dx + z\ dx \wedge dy$$

Then in cartesian coordinates we have

$$d\omega = 3\ dx \wedge dy \wedge dz$$

and in spherical coordinates,

$$d\omega = 3\rho^2 \sin\phi\ d\rho \wedge d\phi \wedge d\theta$$

Proof. In cartesian coordinates, the computation is simple.

$$\begin{aligned} d\omega &= d(x\ dy \wedge dz + y\ dz \wedge dx + z\ dx \wedge dy) \\ &= dx \wedge dy \wedge dz + dy \wedge dz \wedge dx + dz \wedge dx \wedge dy \\ &= 3\ dx \wedge dy \wedge dz \end{aligned}$$

since each of the far right terms can be transformed into $dx \wedge dy \wedge dz$ by performing two swaps. Each swap introduces a negative sign, so the terms remain positive. In spherical coordinates,

$$\begin{aligned} d\omega &= d(\rho^3 \sin\phi\ d\phi \wedge d\theta) = d(\rho^3 \sin\phi) \wedge d\phi \wedge d\theta \\ &= (3\rho^2 \sin\phi\ d\rho + \rho^3 \cos\phi\ d\phi) \wedge d\phi \wedge d\theta = 3\rho^2 \sin\phi\ d\rho \wedge d\phi \wedge d\theta \end{aligned}$$

Now one can do a tedious calculation to check that these are, in fact, equal, but I won't type that out. □

Proposition 0.7 (Exercise 14-6c and 14-6d)**.** *Let ω be the 2-form on $\mathbb{R}^3$ defined above. Let (ϕ, θ) be angle coordinates on S^2 in $\mathbb{R}^3$, and let $\iota_{S^2} : S^2 \to \mathbb{R}^3$ be the inclusion map. Then on $(\phi, \theta) \in (0, \pi) \times (0, 2\pi)$, we have*

$$\iota_{S^2}^*(\omega) = \sin\phi \; d\phi \wedge d\theta$$

Hence this pullback is never zero.

Proof. Using Lemma 14.16(c), we compute

$$i_{S^2}^*\omega = (\rho^3 \sin\phi \circ \iota_{S^2} \; d(\phi \circ \iota) \wedge d(\theta \circ \iota) = \sin\phi \; d\phi \wedge d\theta$$

since $\rho = 1$ on S^2. As this is defined for $\phi \in (0, \pi)$, it is nevery zero since $\sin\phi \neq 0$ on $(0, \pi)$. □

Proposition 0.8 (Exercise 14-7c)**.** *Let $M = \{(u, v) \in \mathbb{R}^2 : u^2 + v^2 < 1\}$ and $N = \mathbb{R}^3 \setminus \{0\}$. Define $F : M \to N$ by*

$$F(u, v) = (u, v, (1 - u^2 - v^2)^{1/2})$$

and define

$$\omega = \frac{x \; dy \wedge dz + y \; dz \wedge dx + z \; dx \wedge dy}{(x^2 + y^2 + z^2)^{3/2}}$$

Then

$$d\omega = 0$$
$$F^*\omega = (1 - u^2 - v^2)^{-1/2} du \wedge dv$$

and we verify by direct computation that $d(F^*\omega) = F^*(d\omega) = 0$.

Proof. First we compute $F^*\omega$. As a shorthand, let

$$\omega_1 = \frac{x}{(x^2 + y^2 + z^2)^{3/2}} \qquad \omega_2 = \frac{y}{(x^2 + y^2 + z^2)^{3/2}} \qquad \omega_3 = \frac{z}{(x^2 + y^2 + z^2)^{3/2}}$$

Then we have

$$\omega_1 \circ F = u \qquad \omega_2 \circ F = v \qquad \omega_3 \circ F = (1 - u^2 - v^2)^{1/2}$$

And we compute

$$d(z \circ F) = d((1 - u^2 - v^2)^{1/2}) = \frac{-u \; du - v \; dv}{(1 - u^2 - v^2)^{1/2}}$$

Then we can compute $F^*\omega$ as

$$\begin{aligned}
F^*\omega &= u \; dv \wedge d(z \circ F) + v \; d(z \circ F) \wedge du + (1 - u^2 - v^2)^{1/2} du \wedge dv \\
&= u \; dv \wedge \left(\frac{-u \; du - v \; dv}{(1 - u^2 - v^2)^{1/2}} \right) + v \; \left(\frac{-u \; du - v \; dv}{(1 - u^2 - v^2)^{1/2}} \right) \wedge du + (1 - u^2 - v^2)^{1/2} du \wedge dv \\
&= \frac{-u^2 \; dv \wedge du}{(1 - u^2 - v^2)^{1/2}} + \frac{-v^2 \; dv \wedge du}{(1 - u^2 - v^2)^{1/2}} + (1 - u^2 - v^2)^{1/2} du \wedge dv \\
&= \frac{u^2 + v^2 + (1 - u^2 - v^2)}{(1 - u^2 - v^2)^{1/2}} du \wedge dv \\
&= \frac{1}{(1 - u^2 - v^2)^{1/2}} du \wedge dv \\
&= (1 - u^2 - v^2)^{-1/2} du \wedge dv
\end{aligned}$$

Now we compute $d\omega$.

$$
\begin{aligned}
d\omega &= \frac{\partial \omega_1}{\partial x} dx \wedge dy \wedge dz + \frac{\partial \omega_2}{\partial y} dx \wedge dy \wedge dz + \frac{\partial \omega_3}{\partial z} dx \wedge dy \wedge dz \\
&= \left(\frac{\partial \omega_1}{\partial x} + \frac{\partial \omega_2}{\partial y} + \frac{\partial \omega_3}{\partial z} \right) dx \wedge dy \wedge dz \\
&= \left(\frac{-2x^2 + y^2 + z^2}{(x^2 + y^2 + z^2)^{5/2}} + \frac{x^2 - 2y^2 + z^2}{(x^2 + y^2 + z^2)^{5/2}} + \frac{x^2 + y^2 - 2z^2}{(x^2 + y^2 + z^2)^{5/2}} \right) dx \wedge dy \wedge dz \\
&= \left(\frac{0}{(x^2 + y^2 + z^2)^{5/2}} \right) dx \wedge dy \wedge dz \\
&= 0
\end{aligned}
$$

We now verify by direct computation that $d(F^*\omega) = F^*(d\omega)$. First we compute $d(F^*\omega)$.

$$
d(F^*\omega) = d((1 - u^2 - v^2)^{-1/2} du \wedge dv) = d((1 - u^2 - v^2)^{-1/2}) \wedge du \wedge dv = 0
$$

since every term has a repeated du or dv. And since $d\omega = 0$, it is obvious that $F^*(d\omega) = 0$ (because F^* is linear). □